

# On the number of Knight's Tours

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Consider the classical problem of the Knight's tour: *find an initial square in an  $8 \times 8$  chess board and a sequence of 63 movements for a knight such that the knight visits every square exactly once*. If we add the restriction that from the last square in the tour the knight should be able to reach the starting square, we have a *re-entrant solution*. In this note we prove that the total number of solutions of this problem, denoted by  $s$ , is divisible by 8 and give a rough upper bound of this quantity.

Also of interest is the number of re-entrant solutions. Each re-entrant solution is naturally associated with other 63 re-entrant solutions, obtained beginning in the second, third,  $\dots$  63<sup>th</sup> square of the given solution, making the corresponding movements in order to have the same sequence of squares. Noting that each of the 64 re-entrant solutions obtained gives a different re-entrant solution inverting the order, we obtain 128 re-entrant solutions.

In terms of graph theory ([1]), if we say that (i) a *path* is a sequence of different squares visited by a knight, (ii) a path is *closed* when from the last square of the sequence the knight can go to the first one, (iii) a path is *Hamiltonian* when it includes all the squares; and observe that we are counting *directed* paths, in view that we consider as different solutions to our problem the same sequence of squares visited in the direct and inverse order, it can be said that  $s$  is the number of *directed Hamiltonian paths*.

The Knight's tour problem has a very long history. Before computers, finding solutions and describing its properties was a challenge attracting the attention of many relevant mathematicians, including Euler, Legendre and Vandermonde [2], [3]. In present times, this well known problem is chosen in order to test computational tools for counting problems with the help of computers. As an example, I. Wegener (in [4]) and Mc Kay (see comments to [5]) obtained independently that the number of classes of re-entrant solutions (not taking into account the order and the initial square) is  $w = 13.267.364.410.532$ .

The divisibility property that we present is proved elementary by considering symmetries and rotations of the chessboard. Denote by  $s(ij)$  be the number of tours beginning in the square  $(i, j)$  ( $i, j = 1, 2, \dots, 8$ ). It is clear that  $s = \sum_{i,j=1}^8 s(ij)$ . Based on invariance under rotations of the chessboard we obtain that

$$s = 4 \sum_{i,j=1}^4 s(ij).$$

Symmetry with respect to the diagonal from  $(1, 1)$  to  $(8, 8)$ , gives that  $s(ij) = s(ji)$  ( $i, j = 1, 2, 3, 4$ ), and we conclude that

$$s = 4\{s(11)+s(22)+s(33)+s(44)+2[s(12)+s(13)+s(14)+s(23)+s(24)+s(34)]\}. \quad (1)$$

In this way, the computation of the total number of tours is reduced to the computation of tours beginning in the squares indicated in Figure 1. It remains

			44				
		33	34				
	22	23	24				
11	12	13	14				

Figure 1: Relevant initial squares

to see that  $s(11), s(22), s(33)$  and  $s(44)$  are even numbers. Denote  $s(ij, kl)$  the number of tours beginning in  $(i, j)$  with first visit to  $(k, l)$ . The same symmetry argument gives

$$\begin{aligned} s(11) &= 2s(11, 23), \\ s(22) &= 2[s(22, 14) + s(22, 34)] \\ s(33) &= 2[s(33, 12) + s(33, 14) + s(33, 25) + s(33, 45)], \\ s(44) &= 2[s(44, 23) + s(44, 25) + s(44, 36) + s(44, 56)]. \end{aligned}$$

In conclusion

$$\begin{aligned} s &= 8[s(11, 23) + s(22, 14) + s(22, 34) + s(33, 12) + s(33, 14) \\ &+ s(33, 25) + s(33, 45) + s(44, 23) + s(44, 25) + s(44, 36) + s(44, 56) \\ &+ s(12) + s(13) + s(14) + s(23) + s(24) + s(34)], \end{aligned}$$

concluding the proof.

Let us add some comments on upper bounds for  $s$ . The computation of  $w$  gives that  $128w \sim 1.68 \times 10^{15}$  is a lower bound for  $s$ .

As the number of possible movements for the knight is 168, and a (non-directed) solution is a set of 63 different movements, we obtain that  $2 \times \binom{168}{63}$  is an upper bound for  $s$  (see [2]). This is approximately  $2.35 \times 10^{47}$ . Another rough upper bound can be obtained as the product of the 64 initial squares by the number of possible movements from each square within a tour, giving  $64 \cdot 2^4 3^8 4^{20} 6^{16} 8^{16}$ . This gives approximately  $5.86 \times 10^{45}$ . If we take into account

the partition of the set of solutions introduced above, and the following facts: (a) excepting in the first case, the number of possible movements from a square is the total of accessible squares (2,3,4,6 or 8) minus one. (b) each corner (i.e. (1, 8)) is connected with two squares with 6 possible movements (in this case (2, 6) and (3, 7)) but, as the solution must visit the corner only one 5 connected to each corner must be counted (excepting solutions beginning in (1, 1)). (c) For the last three movements there are at most two possibilities. This means that the last four factors in the product can be replaced by 1, or by 2 when the corresponding four squares form a closed path. The case with minimum product (taking into account (b)) is 3-2-1-1 in a corner. All other cases gives bigger product, and the closed paths give products bigger than 12. Denote then

$$k = 1^{10}2^73^{19}5^{12}7^{16}.$$

Taking into account the initial squares, and noting that for  $s(11)$  the 1 in (3, 2)

<b>1</b>	<b>1</b>	3	3	3	3	2	1
2	3	<b>1</b>	5	5	5	3	2
<b>1</b>	5	7	7	7	7	1	3
3	5	7	7	7	7	5	3
3	5	7	7	7	7	5	3
3	1	7	7	7	7	5	3
2	3	5	5	5	1	3	2
1	2	3	3	3	3	2	1

Figure 2: Numbers of accessible squares within the tour. 1 and 5 can be interchanged in squares connected with each corner. In bold are indicated four connected squares giving the minimum possible product (then replaced by 1).

must be replaced by 4 possible movements we obtain the following bounds:

$$\begin{aligned} s(11) &\leq 8k & s(22) &\leq 4k/3 \\ s(33) &\leq 8k/7 & s(44) &\leq 8k/7 \\ s(12) &\leq 3k/2 & s(13) &\leq 4k/3 \\ s(14) &\leq 4k/3 & s(23) &\leq 6k/5 \\ s(24) &\leq 6k/5 & s(34) &\leq 8k/7 \end{aligned}$$

Summing up in (1), we obtain  $s \leq \frac{11356}{105}k$ . This is approximately  $1.305 \times 10^{35}$ .

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